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ON FINITE p-GROUPS WHICH HAVE ONLY TWO CONJUGACY LENGTHS

BY

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ABSTRACT

We show that the nilpotent class of any finite group which has only two conjugacy lengths is at most 3. This corresponds to a result of Isaacs and Passman for degrees of irreducible characters.

1. Introduction

Let G be a finite group. We denote by $cd(G)$ and $ccl(G)$ the sets of numbers which occur as the degrees of irreducible characters of G and as the lengths of conjugacy classes of G, respectively. Isaacs and Passman prove in [3] that the commutator subgroup of any finite group G with $cd(G) = \{1, m\}$ $(m > 1)$ is abelian. The conjugacy length version of this result is conjectured by Huppert (p. 445 [1]) and Mann (p. 371 [6]). In this paper we will prove a stronger analogous result for the set of conjugacy lengths. The class of a group G with $ccl(G) = \{1, m\}$ $(m > 1)$ has been introduced by Ito in [5] and he shows that the study of such groups is reduced to that of p-groups for some prime p. Moreover, a result of Isaacs [2] follows, that the central factor of any group of this class is of exponent p. Therefore, any 2-group of this class is of nilpotent class 2. We show the following theorem:

MAIN THEOREM: *Let G be a finite p-group for a prime p such that $ccl(G) = \{1, p^n\}$ $(n \geq 1)$. Then the nilpotent class of G is at most* 3.

The corollary of Isaacs's result (Corollary 2.5 [7]) gives the following.

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COROLLARY: *Let G be a finite p-group for a prime p such that $ccl(G) = \{1, p^n\}$ $(n \geq 1)$. Then the commutator subgroup of G is an elementary abelian p-group.*

2. Main theorem

Let G be a finite group. We define $[x, y] := x^{-1}y^{-1}xy$ and $[x, y, z] := [[x, y], z]$ for all $x, y, z \in G$. The finite field of p elements will be denoted by F_p. The lower central series of G will be denoted by $G_1 \leq G_2 \leq \cdots$, namely $G_1 := G, G_2 := [G, G]$ and $G_{i+1} := [G_i, G]$ $(i \geq 2)$. If $c+1$ is the least value of m satisfying $G_m = 1$, then c is called the nilpotent class of G. The nilpotency class of nilpotent group H will be denoted by $c(H)$.

The following result of Isaacs gives some useful corollaries.

THEOREM 2.1 (Isaacs [2]): *Let G be a finite group, which contains a proper normal subgroup N such that all of the conjugacy classes of G which lie outside of N have the same lengths. Then either G/N is cyclic, or else every nonidentity element of G/N has prime order.*

COROLLARY 2.2: *Let G be a finite p-group such that $ccl(G) = \{1, p^n\}$ $(n \geq 1)$. Then $G/Z(G)$ is of exponent p.*

Note that $G_{i-1}Z(G)/G_iZ(G)$ is an elementary abelian p-group for $2 \leq i \leq c(G)$.

COROLLARY 2.3: *Let G be a finite 2-group such that $ccl(G) = \{1, 2^n\}$ $(n \geq 1)$. Then G is of nilpotent class 2.*

COROLLARY 2.4 (Verardi, Corollary 2.5 [7]): *Let p be an odd prime. Let G be a finite p-group such that $ccl(G) = \{1, p^n\}$ $(n \geq 1)$ and $c(G) \geq 3$. Then $G_{c(G)-1}$ is an elementary abelian p-group.*

Proof of Main Theorem: Suppose that p is an odd prime and $c = c(G) \geq 3$. Then there exist $y \in G$ and $z \in G_{c-2}$ such that $[y, z] \in G_{c-1} - Z(G)$. Now we put $x_1 := [y, z]$, $p^m := |G_{c-1}Z(G)/Z(G)|$ and $G_{c-1}Z(G)/Z(G) = \langle x_1, x_2, \ldots, x_m \rangle Z(G)/Z(G)$. Note that this is an elementary abelian p-group. We denote a coset decomposition of G by $C_G(x_1)$ as the following:

$$G = \bigcup_{\alpha_i=0}^{p-1} u_1^{\alpha_1} u_2^{\alpha_2} \cdots u_n^{\alpha_n} C_G(x_1),$$

and $c_i := [x_1, u_i] \in Z(G)$ for $1 \leq i \leq n$. Since G_{c-1} is a elementary abelian p-group, $|\langle c_1, \ldots, c_n \rangle| = p^n$. By Witt's identity, we have

$$[y, z, u_i][u_i, y, z][z, u_i, y] = 1.$$

Then $[y, z, u_i] = c_i$ and we can write

$$[z, u_i] = x_1^{\beta_{i1}} x_2^{\beta_{i2}} \cdots x_m^{\beta_{im}} w \in G_{c-1}$$

for some $\beta_{ij} \in F_p$ $(1 \leq i \leq n, 1 \leq j \leq m)$ and some $w \in Z(G)$. We put $d_i := [x_i, y]$; then

$$[z, u_i, y] = d_1^{\beta_{i1}} d_2^{\beta_{i2}} \cdots d_m^{\beta_{im}}$$

for $1 \leq i \leq n$. Therefore we have

$$[u_i, y, z] = c_i^{-1} d_1^{-\beta_{i1}} d_2^{-\beta_{i2}} \cdots d_m^{-\beta_{im}}$$

for $1 \leq i \leq n$.

We put $p^s = |\langle c_i^{-1} d_1^{-\beta_{i1}} d_2^{-\beta_{i2}} \cdots d_m^{-\beta_{im}}; 1 \leq i \leq n \rangle|$ and $p^t = |\langle d_1, d_2, \ldots, d_m \rangle|$. Then, since

$$\langle c_1, c_2, \ldots, c_n \rangle \subset \langle d_1, d_2, \ldots, d_m, c_i^{-1} d_1^{-\beta_{i1}} d_2^{-\beta_{i2}} \cdots d_m^{-\beta_{im}}; 1 \leq i \leq n \rangle,$$

we have $s + t \geq n$. We choose $I = \{i_1, i_2, \ldots, i_s\} \subset \{1, 2, \ldots, n\}$ and $J = \{j_1, j_2, \ldots, j_t\} \subset \{1, 2, \ldots, m\}$ such that $|\langle c_i^{-1} d_1^{-\beta_{i1}} d_2^{-\beta_{i2}} \cdots d_m^{-\beta_{im}}; i \in I \rangle| = p^s$ and $|\langle d_{j_1}, \ldots, d_{j_t} \rangle| = p^t$, respectively. Now we consider the subset

$$\bigcup_{\gamma_i = 0}^{p-1} C_G(y) u_{i_1}^{\gamma_1} u_{i_2}^{\gamma_2} \cdots u_{i_s}^{\gamma_s} x_{j_1}^{\gamma_{s+1}} x_{j_2}^{\gamma_{s+2}} \cdots x_{j_t}^{\gamma_{s+t}}$$

of G. We claim that this sum is disjoint. We will show that if

$$[u_{i_1}^{\delta_1} \cdots u_{i_s}^{\delta_s} x_{j_1}^{\delta_{s+1}} \cdots x_{j_t}^{\delta_{s+t}}, y] = [u_{i_1}^{\varepsilon_1} \cdots u_{i_s}^{\varepsilon_s} x_{j_1}^{\varepsilon_{s+1}} \cdots x_{j_t}^{\varepsilon_{s+t}}, y],$$

then $\delta_i = \varepsilon_i$ for $1 \leq i \leq s + t$. This equation is rewritten as follows:

$$[u_{i_1}^{\delta_1} \cdots u_{i_s}^{\delta_s}, y][x_{j_1}^{\delta_{s+1}} \cdots x_{j_t}^{\delta_{s+t}}, y] = [u_{i_1}^{\varepsilon_1} \cdots u_{i_s}^{\varepsilon_s}, y][x_{j_1}^{\varepsilon_{s+1}} \cdots x_{j_t}^{\varepsilon_{s+t}}, y].$$

Then

$$\begin{aligned} [u_{i_1}^{\delta_1} \cdots u_{i_s}^{\delta_s}, y][x_{j_1}^{\delta_{s+1}} \cdots x_{j_t}^{\delta_{s+t}}, y]G_3 &= [u_{i_1}^{\varepsilon_1} \cdots u_{i_s}^{\varepsilon_s}, y][x_{j_1}^{\varepsilon_{s+1}} \cdots x_{j_t}^{\varepsilon_{s+t}}, y]G_3, \\ [u_{i_1}^{\delta_1} \cdots u_{i_s}^{\delta_s}, y]G_3 &= [u_{i_1}^{\varepsilon_1} \cdots u_{i_s}^{\varepsilon_s}, y]G_3, \\ [u_{i_1}^{\delta_1}, y] \cdots [u_{i_s}^{\delta_s}, y]G_3 &= [u_{i_1}^{\varepsilon_1}, y] \cdots [u_{i_s}^{\varepsilon_s}, y]G_3, \\ G_3 &= [u_{i_1}, y]^{\varepsilon_1 - \delta_1} \cdots [u_{i_s}, y]^{\varepsilon_s - \delta_s} G_3. \end{aligned}$$

By $[G_3, z] = 1$, we have

$$\begin{aligned}1 =&[[u_{i_1}, y]^{\varepsilon_1-\delta_1} \cdots [u_{i_s}, y]^{\varepsilon_s-\delta_s}, z]\\ =&\prod_{k=1}^{s}(c_{i_k}^{-1} d_1^{-\beta_{i_k 1}} d_2^{-\beta_{i_k 2}} \cdots d_m^{-\beta_{i_k m}})^{\varepsilon_k-\delta_k}.\end{aligned}$$

By the choice of I, we have $\delta_i = \varepsilon_i$ for $1 \le i \le s$. So it is enough to verify

$$[x_{j_1}^{\delta_{s+1}} \cdots x_{j_t}^{\delta_{s+t}}, y] = [x_{j_1}^{\varepsilon_{s+1}} \cdots x_{j_t}^{\varepsilon_{s+t}}, y].$$

Now

$$\begin{aligned}[x_{j_1}^{\delta_{s+1}} \cdots x_{j_t}^{\delta_{s+t}}, y] =&[x_{j_1}^{\varepsilon_{s+1}} \cdots x_{j_t}^{\varepsilon_{s+t}}, y],\\ [x_{j_1}, y]^{\delta_{s+1}} \cdots [x_{j_t}, y]^{\delta_{s+t}} =&[x_{j_1}, y]^{\varepsilon_{s+1}} \cdots [x_{j_t}, y]^{\varepsilon_{s+t}},\\ d_{j_1}^{\delta_{s+1}} \cdots d_{j_t}^{\delta_{s+t}} =&d_{j_1}^{\varepsilon_{s+1}} \cdots d_{j_t}^{\varepsilon_{s+t}},\\ d_{j_1}^{\delta_{s+1}-\varepsilon_{s+1}} \cdots d_{j_t}^{\delta_{s+t}-\varepsilon_{s+t}} =&1.\end{aligned}$$

Hence, by the choice of J, we have $\delta_i = \varepsilon_i$ for $s+1 \le i \le s+t$. Therefore our claim now follows. Then, by $|G : C_G(y)| = p^n$ and $s+t \ge n$, we have $s+t = n$. Therefore

$$\bigcup_{\gamma_i=0}^{p-1} C_G(y) u_{i_1}^{\gamma_1} u_{i_2}^{\gamma_2} \cdots u_{i_s}^{\gamma_s} x_{j_1}^{\gamma_{s+1}} x_{j_2}^{\gamma_{s+2}} \cdots x_{j_t}^{\gamma_{s+t}}$$

is a right coset decomposition of G by $C_G(y)$. Then we consider the following:

$$[u_{i_1}^{\gamma_1} u_{i_2}^{\gamma_2} \cdots u_{i_s}^{\gamma_s} x_{j_1}^{\gamma_{s+1}} x_{j_2}^{\gamma_{s+2}} \cdots x_{j_t}^{\gamma_{s+t}}, y]$$

for $\gamma_i \in F_p$ $(1 \le i \le s+t)$. By the choice of I, if $(\gamma_1, \ldots, \gamma_s) \ne (0, \ldots, 0)$, then this is contained in $G_2 - G_3$, or else in $Z(G)$. Hence $\{[g, y]; g \in G\} \subset (G_2 - G_3) \cup Z(G)$. Therefore, by $[y, z] = x_1 \in G_{c-1} - Z(G)$, we have $c = 3$. The proof is completed. ∎

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